MR4734698 05C15 68R10

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Trails in arc-colored digraphs avoiding forbidden transitions. (English. English summary)

Discrete Math. Lett. **13** (2024), 6–12.

Let D be a loopless digraph, and suppose we have a coloring of the arcs of D by the vertices of some digraph H, that is, a function $c\colon A(D) \to V(H)$. We call D an *H-colored digraph*. A directed walk $W = (v_0, e_0, v_1, \ldots, v_{k-1}, e_{k-1}, v_k)$ in D is called an *H-walk* if $(c(e_i), c(e_{i+1}))$ is an arc in H for $i = 0, 1, \ldots, k-2$. One can similarly define *H-trails*, *H-paths*, *H-circuits*, and *H-cycles*.

In this paper, the authors give polynomial time algorithms for finding in any H-colored digraph D an H-trail between any two arcs, and between any two vertices, as well as finding such H-trails that are shortest, whenever they exist. For proving these results, it is useful to construct an auxiliary graph as follows. For a vertex u in D, define D_u to be the digraph whose vertices are the arcs of D that are incident with u, and put an arc from e to g if $e = (x, u)$ and $g = (u, y)$ for some vertices x and y in D, and $(c(e), c(g))$ is an arc in H. Then, define the auxiliary graph $L_2^H(D)$ as follows: take the disjoint union of D_u over all the vertices u in D, for each vertex u in D relabel each vertex e in D_u to $f(e, u)$, and lastly add arcs from $f(e, x)$ to $f(e, y)$ for every arc $e = (x, y)$ in D.

In [H. Galeana-Sánchez and C. Vilchis-Alfaro, "Characterizing arc-colored digraphs with an Eulerian trail with restrictions in the color transitions", preprint, 2023, doi:10.2139/ssrn.4474200] it was shown that there is a bijection between the set of H-circuits in D and the set of directed cycles in $L_2^H(D)$. In this paper, it is shown that if $T = (f(e_1, x_1), f(e_2, x_2), \ldots, f(e_k, x_k))$ is a path in $L_2^H(D)$ where $x_1 \neq x_2$ and $x_{k-1} \neq x_k$, then $(x_1, e_1, x_2, e_3, x_4, e_5, \ldots, x_{k-2}, e_{k-1}, x_k)$ is an H-walk in D (it turns out that k must be even under the given conditions).

The authors also prove an analogue of Menger's theorem for H-trails in an H-colored digraph D. For vertices s and t in D, a subset X of arcs of D is called an *(s, t)-H-trails-separator by arcs* if $D - X$ has no H-trail from s to t. Then, their result states that the maximum number of arc-disjoint H-trails in D from s to t is equal to the minimum number of arcs in an (s, t)-H-trails-separator by arcs in D. As a corollary, there is a polynomial time algorithm for maximizing the number of arc-disjoint H-trails between any two vertices.

One can also readily specialize these results to the usual notion of r-arc-colored digraphs, $r \in \mathbb{N}$, by taking H to be a loopless complete digraph. In particular, it is observed that the problem of deciding whether an H-colored digraph D contains an H-path between two vertices is NP-complete, since the problem is NP-complete even over r-arc-colored digraphs [see L. Gourvès et al., Discrete Appl. Math. **161** (2013), no. 6, 819–828; MR3027971]. Also, using a characterization due to G. Benítez-Bobadilla, Galeana-Sánchez and C. Hernández-Cruz [Discuss. Math. Graph Theory **44** (2024), no. 2, 519–537; MR4715692], the authors show that if H is a transitive and reflexive digraph, then one can find in an H-colored digraph D an H-path in D (if it exists) between any two vertices in polynomial time. Similarly, one can find in an arc-colored digraph D a monochromatic path in D (if it exists) between any two vertices in polynomial time.

Lastly, the authors show that the results in this paper can be equivalently formulated in terms of T-compatible walks in a digraph D with a transition system T, defined as follows. For each vertex u of D, let $T(u)$ be some set of pairs of arcs (e, g), where $e = (x, u)$ and $g = (u, y)$ for some vertices x and y of D. Then, the members of $T(u)$ are called *permitted transitions*, $T(u)$ is called the *transition digraph at u*, and the collection $T = \{T_u : u \in V(D)\}$ is called a *transition system*. A walk in a digraph D with a transition system T is said to be T-*compatible* if in the walk only permitted transitions occur. It is easy to see that given an H-colored digraph D one can obtain a transition system T on the digraph D, and vice-versa; moreover, H-walks in an H-colored digraph D are precisely T-compatible walks in the digraph D equipped with the corresponding transition system T. The authors pose the following open problem: given a digraph D with transition system T, find a digraph H with the minimum number of vertices such that there exists an H-coloring of D with the property that H-walks in D and T-compatible walks in D coincide.

A few typos:

- In the paragraph preceding the statement of Theorem 1.6, it should read "we add a directed path from $f(e,x)$ to $f(e,y)$" instead of "$\ldots$ from $f(e,u)$ to $f(e,v)$".
- In Observation 2.1(b), the definitions of A_u^+ and A_u^- are swapped.
- In the statement of Lemma 2.1, it should say "H-trail in D" instead of "H-trail in G". Similarly, in the second line of the proof of Lemma 2.1, it should say "$(f(e_1,x_1), f(e_2,x_2)) \notin A(D_{x_1})$" instead of "$\ldots \notin A(G_{x_1})$".

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